MATH6041
Some Topics in Hyperbolic Conservatin Laws

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Course Contents:

- Introduction to shock wave theory, I (General)
- Introduction through the Burgers Equation, II
- Theory for general scalar conservation laws (Krushkov theory and compensated compactness)
- Riemann Problems (Lax Theory)
- Glimm Theory and Random Choice method

- Front tracking method and Continuous dependence
- Uniqueness of viscosity solutions
- Large time asymptotic behavior of solutions
- Selected topics in multi-dimensional theory:
  - Non-uniqueness results;
  - Singularity-formation;
  - Boundary-value problems.

References:

1. J. A. Smoller: Shock Waves and Reaction-Diffusion Equations, Spring-Verlag
2. C. M. Dafermos: Hyperbolic Conservation Laws in Continuous Mechanics
3. Alberto Bressan: Hyperbolic Systems of Conservation Laws: One-Dimensional Cauchy Problem
4. A. Majda: Compressible Fluid Flow and System of Conservation Laws in Several Space Variables
5. D. Serre: Systems of Conservation Laws: Vols 1&2, 1999.

Assessment of Scheme:
Each student will be asked to write a report on a topic assigned by the teacher (the topic can be a paper related to the course or a report o the content learned in the class).

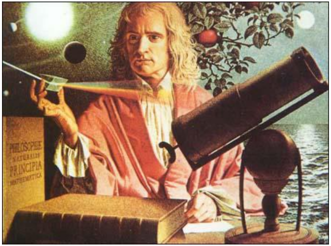
An Overview of Shock Wave Theory

September 8, 2020

The Chinese University of Hong Kong

Contents

§1. Compressible flows and Euler Systems



Newton (1642-1727)

Euler (1707-1783)

§1. Compressible flows and Euler Systems

The macro-scopic description of compressible fluids in Newtonian mechanic is given by

$$
\text{Euler system} \begin{cases}
\partial_t \rho + \operatorname{div}(\rho \vec{u}) = 0 \\
\qquad \text{(conservation of mass)} \\[2ex]
\partial_t(\rho \vec{u}) + \operatorname{div}(\rho \vec{u} \times \vec{u}) + \nabla p = \rho F \\
\qquad \text{(conservation of momentum)} \\[2ex]
\partial_t(\rho E) + \operatorname{div}(\rho \vec{u} E + \rho \vec{u}) = \rho F \cdot \vec{u} \\
\qquad \text{(conservation of energy)}
\end{cases}
$$

where:

$$\begin{aligned} &\rho : \text{density} \\ &\vec{u} : \text{velocity} \\ &e : \text{internal energy} \\ &E : \text{total energy} \\ &p : \text{pressure} \\ &F : \text{external force} \end{aligned}$$

Equation of state: $p = p(e, \rho)$

§1. Compressible flows and Euler Systems

This is the well-known compressible Euler system, which is one of the most important systems of hyperbolic conservation laws. One of the most fundamental issues in the mathematical theory of macro-scopic compressible fluids is to understand the global well-posedness of solutions to the Euler system. This has been the driving force for the mathematical theory of nonlinear hyperbolic conservation laws, which has remained to be one of the biggest challenges in the theory of nonlinear PDEs especially in multi-space dimensions.

Basic Features: Discovery of Bernard Riemann

The speeds of the propagation of a sound wave with finite amplitude depend on the sound wave itself $\Rightarrow$

Formation of Shock Waves & Rarefaction Waves

This forces one to deal with the following issues:

- Weak solutions;
- Uniqueness and entropy conditions;
- Dissipations and wave interactions;
- Regularities and structure of solutions;
- etc.

Some Simplified Models of Euler Equations

(1) Planary waves by weakly nonlinear geometric optics $\Rightarrow$ Burgers equation

$$\partial_t u + \partial_x\left(\frac{u^2}{2}\right) = 0$$

Fact: Many important nonlinear phenomena can be explained by the Burgers equation!!

(2) Propagations of infinitesimal sound waves $\Rightarrow$ Wave equation

$$\partial_t^2 p - c^2 \Delta p = 0$$

with c being constant.

(3) Steady flow equations

$$
\begin{cases}
\operatorname{div}(\rho\vec{u}) = 0 \\
\operatorname{div}(\rho\vec{u}\otimes\vec{u}) + \nabla p = 0 \\
\operatorname{div}(\rho\vec{u}E + \vec{u}p) = 0
\end{cases}
$$

Basic feature:

- mixed type PDEs;
- change-type and degenerate PDEs.

Two important particular cases:

(3.1) Steady Potential Flow equations:

steady + isentropic + irrotational $\Rightarrow$

$$\vec{u} = \nabla \phi$$

$$\sum_{i=1}^{N} ((\partial_i \varphi)^2 - c^2(|\nabla \phi|))\partial_i^2 \phi + 2 \sum_{1 \leq i \leq j \leq N} \partial_i \varphi \, \partial_j \phi \, \partial_{ij}^2 \phi = 0$$

with $c^2(\rho) = p'(\rho)$ (c: sound speed) and Bernoullis law:
$\frac{1}{2}|\nabla \phi|^2 + h(\rho) = c_0$ with enthalpy h defined as

$$h'(\rho) = \frac{c^2(\rho)}{\rho}$$

Basic Feature:

$$\text{The potential equation is} \begin{cases} \text{hyperbolic} & \text{if} \quad M > 1 \text{ (supersonic)} \\ \text{elliptic} & \text{if} \quad M < 1 \text{ (subsonic)} \\ \text{parabolic} & \text{if} \quad M = 1 \text{ (sonic)} \end{cases}$$

here $M = \dfrac{|\vec{u}|}{c}$ is the Mach number of the flow.

Remark:

This is one of the most interesting change-type degenerate PDEs.

(3.2) Two-dimensional isentropic steady flows:

$$\begin{cases} \partial_x(\rho u) + \partial_y(\rho v) = 0 \\ \partial_x(\rho u^2) + \partial_y(\rho uv) + \partial_x p = 0 \\ \partial_x(\rho uv) + \partial_y(\rho v^2) + \partial_y p = 0 \end{cases}$$

The characteristic speeds of this system are: $\lambda_1 = \dfrac{v}{u}$,

$$\lambda_\pm = \frac{uv \pm c(\rho)\sqrt{u^2 + v^2 - c^2(\rho)}}{u^2 - c^2}$$

$$\text{The system is } \begin{cases} \underline{\text{hyperbolic}} & \text{if} \quad M > 1 \text{ (supersonic)} \\ \underline{\text{hyperbolic}} + \underline{\text{elliptic}} & \text{if} \quad M < 1 \text{ (subsonic)} \\ \underline{\text{degenerate}} & \text{if} \quad M = 1 \text{ (sonic)} \end{cases}$$

(4) Incompressible Euler System

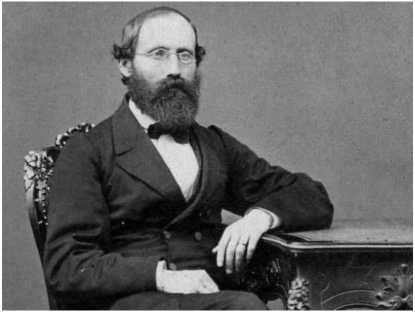
As the Mach number $M = \dfrac{|\vec{u}|}{c} \ll 1$, then the compressible Euler system is approximated by the incompressible Euler system

$$\begin{cases} \partial_t \vec{u} + \operatorname{div}(\vec{u} \otimes \vec{u}) + \nabla p = 0 \\ \operatorname{div} \vec{u} = 0 \end{cases}$$

which is the basic model for incompressible fluid dynamics:

- hyperbolic-elliptic coupled system;
- 2-dimension theory is satisfactory;
- 3-D global well-posedness theory is open.

§2. Contributions of Riemann and Riemann Problems



Bernhard Riemann (1826-1866)

§2. Contributions of Riemann and Riemann Problems

Before Riemann, most scientists studied the propagation of infinitesimal sound waves, which are governed by the linear wave equation, which says that the sound waves propagates on characteristic surfaces with constant speeds. Thus the form of solutions will not change and there is no finite time singularity formation from a smooth initial profile. In his 1860 paper: *Verber die Fortflazung ebener Luftwellen von endlicher Schwingunqsweite, Gött. Abh. Math. C1, 8(1860), 43-65.* Riemann was the first one to study the propagation of sound waves of finite amplitudes.

Riemann's Discoveries

An initial distance of finite amplitude splits two opposite waves with moving speed $u + c(\rho)$ and $u - c(\rho)$ respectively, where u is the fluid velocity and $c(\rho)$ the local sound speed. Furthermore, the dependence of the propagation speed $u \pm c(\rho)$ on the density leads to the compression and expansion of the sound waves. These yield formation of shock waves and rarefaction waves. He also discovered Riemann Invariants.

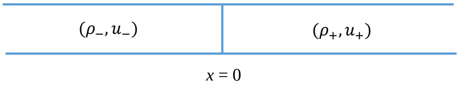
In fact, Riemann solved the following one-dimensional shock-tube problem for the one-dimensional isentropic Euler system:

§2. Contributions of Riemann and Riemann Problems

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p(\rho)) = 0 \end{cases}$$

$$(\rho, u)(t = 0, x) = \begin{cases} (\rho_-, u_-), & x < 0 \\ (\varphi_+, u_+), & x > 0 \end{cases}$$

where (ρ_-, u_-) and (ρ_+, u_+) are constant states.



Riemann Problem

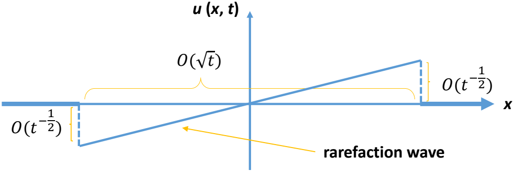
§2. Contributions of Riemann and Riemann Problems

<u>Remark</u>: Since the compressible Euler system is dialation invariant, so the solutions to the Riemann Problem are self-similar solutions of the compressible Euler equations. In 1-dimension, such solutions are not only building blocks of general solutions, but also govern both local and large time behavior of physical weak solutions. So Riemann's discoveries and his problem are essential for the birth of mathematical theory of shock waves for general hyperbolic conservation laws.

§3. Progress in 1-dimensional Space

- B. Riemann (1860): for Isentropic Euler:
  - Riemann Problem;
  - Nonlinear hyperbolic waves.
- Many studies for Riemann Problems for general Euler system and for some practical problems, see Courant-Friedrich's (1948).
- E. Hopf (1950): for Burger's equation:
  - Entropy condition (viscosity criteria);
  - Dissipation mechanism due to nonlinearity
    - $\Rightarrow$ cancellation of shock and rarefaction waves
    - $\Rightarrow$ decay of solutions
    - $\Rightarrow$ N-wave!

§3. Progress in 1-dimensional Space



- Hoft-Lax-Oleinik formula.
- <u>Formation of shocks</u>

  P. Lax, Fritz John, Liu.

§3. Progress in 1-dimensional Space

- Loss of uniqueness and entropy conditions:
  - Viscous criteria (E. Hopf, Oleinik, Gelfand, Smoller, etc.)
  - Lax's geometrical condition $\Leftrightarrow$ structural stability condition
  - Physical entropy criteria: the physical entropy increases across shocks.

    ...

  Fact: All of these criteria are equivalent for weak waves!

- The general Riemann problem is completely solved uniquely by P. Lax in 1957.

- Discovery of the suitable solution space:

  BV: space of bound total variation

  Glimm, Smoller-Conway, Volport, etc.

§3. Progress in 1-dimensional Space

- Global existence of entropy weak solution with small total variation in BV by Glimm in 1968.

Key ideas:

- Riemann solutions are building blocks
- Wave interactions and Glimm's interaction functional $\Rightarrow$ A priori total variation estimates $\Rightarrow$ convergence
- random choice method $\Rightarrow$ consistency

- Structure and Asymptotic behavior of BV solutions (and linear and nonlinear stability of basic waves).

§3. Progress in 1-dimensional Space

Key ideas:

Riemann's Discovery of propagation speeds depending on waves

$+$ entropy conditions

$\Rightarrow$ shock waves must interact with rarefaction waves

$\Rightarrow$ cancellation must occur

$\Rightarrow$ decreasing of wave strength

$\Rightarrow$ decay to Riemann solutions

§3. Progress in 1-dimensional Space

- Glimm-Lax (1971)

$$|u(x,t)-\bar{u}|_{L^\infty[0,T]} \leq \frac{CL}{t}$$

  $u(x,t)$ is periodic weak entropy solution with period L. The fundamental paper along this line.

  Key tools: Riemann Invariants.

- Diperna: structure of BV solutions: see the entropy BV solution locally is a small perturbation of Riemann solution (1970's)
- Diperna, Liu, Dafermos.
- Asymptotic stability of linear and nonlinear waves.

§3. Progress in 1-dimensional Space

Conclusion: The Riemann solutions determine the asymptotic behavior of a general entropy satisfying BV solutions both locally and globally.

- Uniqueness and continuous dependence of viscosity weak solutions, Bressan 2001.

Basic Idea:

- local behavior as Riemann's solution
- weighted comparison nonlinear functional

§3. Progress in 1-dimensional Space

- Viscous approximation of the Euler system, uniform BV estimates and vanishing viscosity limits (artifical viscosities), Bressan-Bianchini, 2003-2009.
- Existence of entropy weak solutions with arbitrary amplitudes to the isentropic Euler equations (2×2 system) and theory of compensated compactness:

  Tartar, Diperna, Serre, Ding, P. L. Lions, $\cdots$

Summary: One-dimensional theory is almost complete and satisfactory.

§3. Progress in 1-dimensional Space

Major open problems in 1-dimension:

1. Global existence and large time behavior of entropy weak solutions to the full Euler system with periodic initial data? See Qu-Xin 2015 (ARMA)
2. Uniform BV estimates for the compressible Navier-Stokes system and vanishing viscosity limit to the Euler equations?
3. Fine regularity and generic structure of general entropy weak solutions for 1-d-systems.

§4. Challenges in Multi-Dimensional Shock Wave Theory

- Theory for scalar equation is complete;
- Local well-posedness of smooth solution based on entropy-variables (Friedriches-Lax, Kato, etc., 1950's); Cauchy problem, IBVP (Kreiss Theory);
- Shock formation: Christodoulou-Miao (2014), Alihanc (1990's); Sideris, etc.;
- Local existence of non-planary elementary waves (1980's, A. Majda, Meltivier, etc.)
- Physically important wave patterns;
- Vacuum dynamics.

§4. Challenges in Multi-Dimensional Shock Wave Theory

Difficulties and Challenges:

- Riemann Problems are not fundamental as in 1-D.
- BV spaces are not suitable for the global existence of entropy weak solutions in Multi-dimensional! (1983, J. Rauch)
- In general, L^∞-entropy weak solutions are not unique:
  - Existence of infinitely many L^∞-entropy weak solutions for the multi-dimensional isentropic Euler system with shock data (De Lellis-Szèkelyhidi (2009-2010), Chiodaroli-De Lellis-Kreml (2014))

§4. Challenges in Multi-Dimensional Shock Wave Theory

- Structure of such "wild" solution. These wild solutions can be highly oscillatory, indeed, such a solution could take only 5 states (in 2-dimensional), and 9 states (in 3-dimensional). So the solution can be everywhere discontinuous (Luo-Xie-Xin, 2015).
- Such wild solutions cannot be <u>ruled out</u> by other physical effects such as rotations or dampings (Luo-Xie-Xin) or heat conduction (Feireisl) or surface tension (Feireisl-Marcati-Tonelli), or even partial viscosities (Luo).

Remarks:

1. Such a wild solution is constructed by the method of convex integration which depends crucially on the lower regularity of the solution space and the high space dimensionality!
2. In L^∞-space, the "wild" solution satisfies most of the known entropy criteria except the "vanishing viscosity criteria".

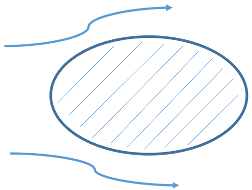
One of the main open questions:

Find a better space (than L^∞), such that the uniqueness is obtained for those weak solutions under only the physical entropy condition.

§4. Challenges in Multi-Dimensional Shock Wave Theory

Some Progress on Special Physically Relevant Flow Patterns

1. Subsonic Flows past a solid body



- irrotation steady flows: almost done!

  Bears, Gilberg, Shiffmann, Dong, $\cdots$, (1950's)
- rotational steady flows: symmetric body (2015),

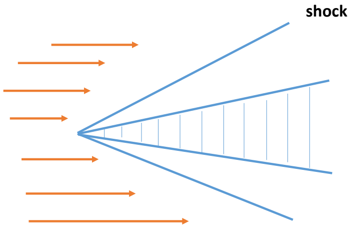
  Chen-Du-Xie-Xin, open in general.

§4. Challenges in Multi-Dimensional Shock Wave Theory

2. Supersonic flows past a solid body

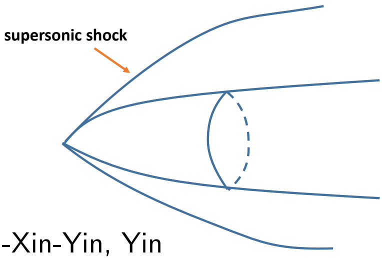
§4. Challenges in Multi-Dimensional Shock Wave Theory

- 2D wedge problem: <u>many results</u>



§4. Challenges in Multi-Dimensional Shock Wave Theory

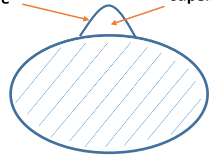
- 3D steady problem: irrotational flow:



  Chen, Chen-Xin-Yin, Yin

- Instability of transonic shock (Yin, Xu-Yin)

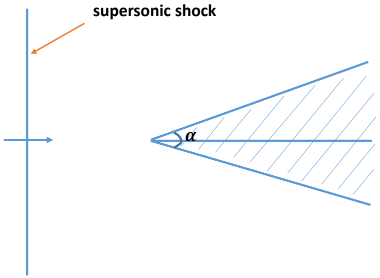
3. Instability of smooth transonic flows



C. Morawetz (1950's): such a wave pattern is unstable!!

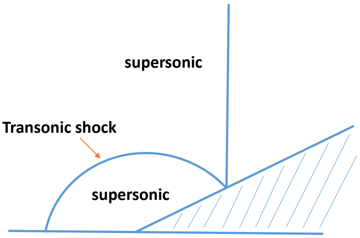
<u>Open Problem</u>: Piecewise smooth transonic flows with shocks past a solid body?

4. Shock Reflection Problems (Riemann Problem)



Various possibilities

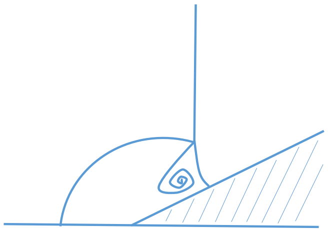
- RR



Von. Neumann (1930's), Courant-Friedrich's (1940's), Morawetz (1980), Chen-Feldman (2010), Valker-Liu (2008).

§4. Challenges in Multi-Dimensional Shock Wave Theory

- MR: open



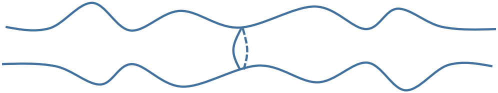
- DMR etc.

§4. Challenges in Multi-Dimensional Shock Wave Theory

<u>Remarks</u>: All the important cases here are open due to

- free-boundaries;
- mixed-type PDE;
- strong degeneracy;
- strong nonlinearities;
- complex geometry, etc.

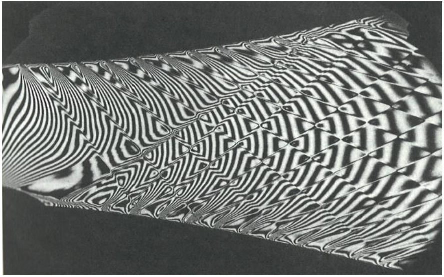
5. Smooth subsonic steady flows in a nozzle



m: mass flux

- irrotational: L. Bers, Xie-Xin (2007)
- rotational: Xie-Xin (2010), etc.

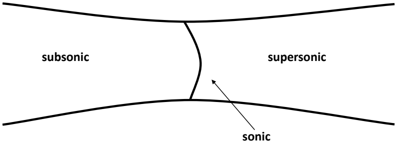
6. Supersonic flows in a nozzle, very complicated reflection patterns



3-d: open!

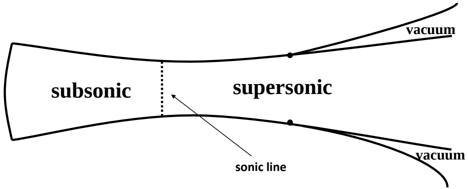
7. Smooth transonic steady flows:

- Meyer type flow

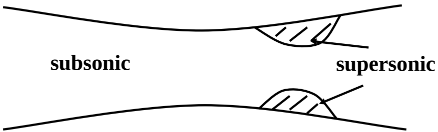


Major difficulties: strong degeneracy at the sound curve which is free in general!

Wang-Xin (2014): irrotational flow

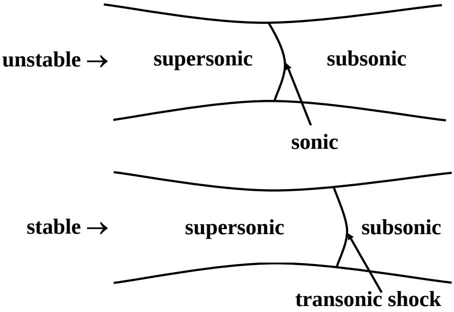


- Taylor's type flows



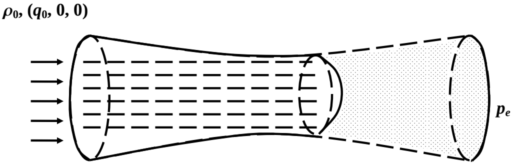
C. Morawetz: such a flow is unstable

How about


unstable →
supersonic
subsonic
sonic
stable →
supersonic
subsonic
transonic shock

8. Transonic shocks in a de Laval nozzle:

   Courant-Friedrich's Problems (1948): Motivated by engineering studies, Courant-Friedrichs proposed the following problem on transonic shock phenomena in a de Laval nozzle:



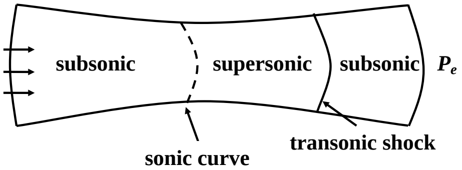
Summary of Major Results:

- Solved for expanding cone by Courant-Friedrich (1948).
- Ill-posedness by potential flows (Xin-Yin, 2005-2007).
- For modified potential flow, the problem is well-posed in 3-D by Bae-Feldman, 2010.
- Completely solved in 2D by Li-Xin-Yin, 2009-2013.
- Dynamical stability for symmetric flows (Xin-Yin, Rauch-Xie-Xin).

Remarks:

- As for the shock reflection problem, all the major difficulties there are present here except the strong degeneracy of sonic state. One of the key difficulties is that in the subsonic region, the governing system is mixed type (elliptic + hyperbolic), so the possible loss of regularity due to hyperbolic models is essential! This is the main reason that the problem is still open in 3-dimension!

- Even in 2-dimension, an important physically interesting pattern is



which is open!!

Thank You!